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Fixed point property for Banach algebras associated to locally compact groups

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Dedicated to our teacher, Professor Edmond E. Granirer, on the occasion of his 75th birthday with admiration and respect

Abstract

In this paper we investigate when various Banach algebras associated to a locally compact group G have the weak or weak* fixed point property for left reversible semigroups. We proved, for example, that if G is a separable locally compact group with a compact neighborhood of the identity invariant under inner automorphisms, then the Fourier–Stieltjes algebra of G has the weak* fixed point property for left reversible semigroups if and only if G is compact. This generalizes a classical result of T.C. Lim for the case when G is the circle group T.

Keywords: Group C^*-algebra; Group von Neumann algebra; Fourier algebra; Fourier–Stieltjes algebra; Weak* uniform Kadec–Klee property; Weak* normal structure; Weak* fixed point property; Nonexpansive mapping; Left reversible semigroup; Commutative semigroup

1. Introduction

Let E be a Banach space and K be a nonempty bounded closed convex subset of E. We say that K has the *fixed point property* if every nonexpansive mapping $T : K \to K$ (i.e. $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K$) has a fixed point. We say that E has the *weak fixed point property* if

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every weakly compact convex subset of E has the fixed point property. A dual Banach space E is said to have the weak* fixed point property if each weak* compact convex subset of E has the fixed point property. As we have a need to refer to results in [26,27] occasionally, we should point out that weak fixed point property and weak* fixed point property in there are denoted by FPP (or fpp) and FPP* (or fpp*), respectively.

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \to as$ and $s \to sa$ from S into S are continuous. S is called *left reversible* if $\overline{aS} \cap \overline{bS} \neq \emptyset$ for any $a, b \in S$, where, in general, $\overline{K}$ denotes the closure of the set K. Clearly abelian semigroups and groups are left reversible. Let $CB(S)$ be the C^*-algebra of bounded continuous complex-valued functions on S and for $a \in S$, let l_a be the left translation operator on $CB(S)$ be defined by $(l_a f)(t) = f(at)$ for all $f \in CB(S)$ and for all $t \in S$. Then S is *left amenable* if there is an $m \in CB(S)^*$ such that $\|m\| = m(1) = 1$ and $m(l_a f) = m(f)$ for all $f \in CB(S)$ and $a \in S$. If the topology on S is normal and S is left amenable, then S is left reversible. In particular, if S is left amenable as a discrete semigroup, then S is left reversible. Left reversible semigroups have played an important role in the study of common fixed point theorems and ergodic type theorems for semigroups of nonexpansive mappings (see [18,22,23,30–32,34–36]).

Let S be a semitopological semigroup, and K be a topological space. An action of S on K is a map ψ from $S \times K$ to K, denoted by $\psi(s,k) = sk$, $s \in S$, $k \in K$, such that $s_1 s_2(k) = s_1(s_2 k)$, for all $s_1, s_2 \in S$, and $k \in K$. The action is separately continuous if ψ is continuous in each of the variables when the other is kept fixed. Lau showed in [22] that if E is a Banach space and $\mathcal{S} = \{T_s\colon\ s \in S\}$ is a continuous representation of a left reversible semitopological semigroup S as nonexpansive self-maps on a compact convex subset K of E, then K contains a common fixed point for $\mathcal{S}$. We say a Banach space E has the *weak fixed point property for left reversible semigroups* if whenever S is a left reversible semitopological semigroup and K is a nonempty weakly compact convex subset of E for which the action of S on K (with the norm topology) is separately continuous and nonexpansive, then K has a common fixed point for S. Similarly a dual Banach space E has the *weak* fixed point property for left reversible semigroups* if whenever S is a left reversible semitopological semigroup and K is a nonempty weak* compact convex subset of E for which the action of S on K is separately continuous and nonexpansive, then K has a common fixed point for S. In general, a weakly compact convex set of a Banach space need not have the fixed point property for left reversible semigroups, not even commutative semigroups. Indeed, Alspach [1] (see also [3, Theorem 4.2], [4,8]) showed there is a weakly compact convex subset K in $L^1[0,1]$ and an isometry $T : K \to K$ without a fixed point. Hence if $S = (\mathbb{N}, +)$ and $\mathcal{S} = \{T^n\colon\ n \in \mathbb{N}\}$, then K does not have a common fixed point for $\mathcal{S}$. However, Bruck showed in [5] that a Banach space E having the weak fixed point property has the weak fixed point property for commutative semigroups, and Lim showed in [34] that a Banach space with weak normal structure has the weak fixed point property for left reversible semigroups. For dual Banach spaces, it is known (see [34,35]) that ℓ_1 and any uniformly convex Banach space have the weak* fixed point property for left reversible semigroups.

This paper is organized as follows. In Section 3, we shall establish a technical lemma that we shall need for our result on weak* fixed point property for left reversible semigroups. In Section 4, we prove our main results concerning the weak* fixed point property for left reversible semigroups on the Fourier–Stieltjes algebra of a locally compact group and its relations with other geometric properties. In Section 5, we shall study the weak fixed point property for left reversible semigroups or commuting semigroups on various Banach algebras associated to a locally compact group. In Section 6, we shall discuss some open problems arising from this work.

2. Some preliminaries

Let K be a bounded closed convex subset of a Banach space E. A point x in K is called a *diametral point* if

$$\sup\{\|x-y\|:\ y \in K\} = \operatorname{diam}(K),$$

where $\operatorname{diam}(K)$ denotes the diameter of K. The set K is said to have *normal structure* if every nontrivial (i.e., contains at least two points) convex subset H of K contains a non-diametral point of H (see [15,20]). A Banach space E has *weak normal structure* if every nontrivial weakly compact convex subset of E has normal structure. A dual Banach space E has *weak* normal structure* if every nontrivial weak* compact convex subset of E has normal structure. A Banach space E is said to have property UKK (*uniformly Kadec–Klee property*) if for any $\varepsilon > 0$ there is a $0 < \delta < 1$ such that whenever (x_n) is a sequence in the unit ball of E converging weakly to x and satisfying $\operatorname{sep}((x_n)) \equiv \inf\{\|x_n - x_m\|:\ n \neq m\} > \varepsilon$, then $\|x\| \leqslant \delta$. A dual Banach space E is said to have property UKK* (*weak* uniformly Kadec–Klee property*) if for any $\varepsilon > 0$ there is a $0 < \delta < 1$ such that whenever A is a subset of the closed unit ball of E containing a sequence (x_n) with $\operatorname{sep}((x_n)) > \varepsilon$, then there is an x in the weak* closure of A such that $\|x\| \leqslant \delta$. The property UKK* was introduced by van Dulst and Sims [11]. They proved that if E has property UKK*, then E has weak* normal structure and hence has the weak* fixed point property.

Let G be a locally compact group with a fixed left Haar measure λ. Let $L^1(G)$ be the group algebra of G with convolution product. We define $C^*(G)$, the group C^*-algebra of G, to be the completion of $L^1(G)$ with respect to the norm

$$\|f\|_* = \sup\|\pi_f\|,$$

where the supremum is taken over all nondegenerate $*$-representations π of $L^1(G)$ as a $*$-algebra of bounded operators on a Hilbert space. Let $\mathcal{B}(L^2(G))$ be the set of all bounded operators on the Hilbert space $L^2(G)$ and ρ be the left regular representation of G, i.e., for each $f \in L^1(G)$, $\rho(f)$ is the bounded operator in $\mathcal{B}(L^2(G))$ defined by $\rho(f)(h) = f * h$, the convolution of f and h in $L^2(G)$. Denote by $C^*_\rho(G)$ the completion of $L^1(G)$ with the norm $\|\rho(f)\|$, $f \in L^1(G)$, and denote by $VN(G)$ the closure of $\{\rho(f):\ f \in L^1(G)\}$ in the weak operator topology in $\mathcal{B}(L^2(G))$. In the case when G is left amenable, which is the case when G is compact, then $C^*(G)$ is isometric isomorphic to $C^*_\rho(G)$. Denote the set of continuous positive definite functions on G by $P(G)$, and the set of continuous functions on G with compact support by $C_{00}(G)$. We define the Fourier–Stieltjes algebra of G, denoted by $B(G)$, to be the linear span of $P(G)$. Then $B(G)$ is a Banach algebra with the norm of each $\phi \in B(G)$ defined by

$$\|\phi\| = \sup_{f \in L^1(G),\ \|f\|_* \leqslant 1} \left| \int f(t)\phi(t)\, d\lambda(t) \right|.$$

The Fourier algebra of G, denoted by $A(G)$, is defined to be the closed linear span of $P(G) \cap C_{00}(G)$. Clearly, $A(G) = B(G)$ when G is compact. It is known that $C^*(G)^* = B(G)$, where the duality is given by $\langle f, \phi\rangle = \int f(t)\phi(t)\, d\lambda(t)$, $f \in L^1(G)$, $\phi \in B(G)$, and $A(G)^* = VN(G)$ (see [13] for details).

3. Technical lemma

In preparation for our results on the weak* fixed point property for left reversible semigroups for $B(G)$, we first establish the following lemma.

Lemma 3.1. *Let G be a compact group, and let $\{D_\alpha\colon \alpha \in \Lambda\}$ be a decreasing net of bounded subsets of $B(G)$, and $\{\phi_m\colon m \in M\}$, be a weak* convergent sequence with weak* limit ϕ. Then*

$$\limsup_m \limsup_\alpha \bigl\{\|\phi_m - \psi\|\colon \psi \in D_\alpha\bigr\} = \limsup_\alpha \bigl\{\|\phi - \psi\|\colon \psi \in D_\alpha\bigr\} + \limsup_m \|\phi_m - \phi\|. \tag{3.1.1}$$

Proof. Since G is compact, it follows from Lemma 4.1 and Theorem 4.2 in [29] that $C^*(G)$ is a c_0-sum of finite-dimensional C^*-algebras. But each finite-dimensional C^*-algebra is a finite direct sum of full matrix algebras [37, Theorem 11.2]. Thus we may write

$$C^*(G) = c_0 - \sum_{i\in I} \bigoplus \mathcal{K}(\mathfrak{H}_i) \quad \text{and} \quad B(G) = l_1 - \sum_{i\in I} \bigoplus \mathcal{T}(\mathfrak{H}_i), \tag{3.1.2}$$

where $\mathcal{K}(\mathfrak{H}_i)$ and $\mathcal{T}(\mathfrak{H}_i)$ are the compact operators and the trace class operators on the finite-dimensional Hilbert space $\mathfrak{H}_i$, respectively. Let $\psi \in B(G)$ and write $\psi = (\psi(i))$, $\psi(i) \in \mathcal{T}(\mathfrak{H}_i)$. Clearly for each $\psi \in D_\alpha$, we have

$$\|\phi - \psi\| + \|\phi_m - \phi\| \geqslant \|\phi_m - \psi\|.$$

Hence if α_0 is fixed, then for each $m \in M$,

$$\sup_{\alpha\geqslant\alpha_0} \bigl\{\|\phi - \psi\|\colon \psi \in D_\alpha\bigr\} + \|\phi_m - \phi\| \geqslant \sup_{\alpha\geqslant\alpha_0} \bigl\{\|\phi_m - \psi\|\colon \psi \in D_\alpha\bigr\},$$

and so

$$\limsup_\alpha \bigl\{\|\phi - \psi\|\colon \psi \in D_\alpha\bigr\} + \|\phi_m - \phi\| \geqslant \limsup_\alpha \bigl\{\|\phi_m - \psi\|\colon \psi \in D_\alpha\bigr\}.$$

Thus the right side is greater or equal to the left side in (3.1.1).

To prove the reverse inequality, we first show that we can assume $\{D_\alpha\colon \alpha \in \Lambda\}$ is a decreasing sequence $\{D_n\colon n \geqslant 1\}$ of bounded sets. For each $\alpha \in \Lambda$, let

$$\rho_\alpha := \sup\bigl\{\|\phi - \psi\|\colon \psi \in D_\alpha\bigr\} \in [0, \infty),$$

and let

$$\rho := \lim_{\alpha\in\Lambda} \rho_\alpha = \inf_{\alpha\in\Lambda} \rho_\alpha \in [0, \infty).$$

For each $k \in \mathbb{N}$, we can choose $\alpha_k \in \Lambda$ such that

$$\rho \leqslant \rho_{\alpha_k} < \rho + \frac{1}{k}.$$

Moreover, we may choose the sequence (α_k) so that $\alpha_{k+1} \geqslant \alpha_k$ for all $k \in \mathbb{N}$. Note that for $\alpha \in \Lambda$ with $\alpha \geqslant \alpha_k$,

$$\rho \leqslant \rho_\alpha \leqslant \rho_{\alpha_k} < \rho + \frac{1}{k}.$$

Thus $\{D_{\alpha_k}\colon k \in \mathbb{N}\}$ is a decreasing sequence and

$$\rho := \lim_{\alpha \in \Lambda} \rho_\alpha = \lim_{k \in \mathbb{N}} \rho_{\alpha_k}.$$

Next, for each $m \in M$ and each $\alpha \in \Lambda$, let

$$\nu_\alpha^m := \sup\bigl\{\|\phi_m - \psi\|\colon \psi \in D_\alpha\bigr\} \in [0, \infty)$$

and

$$\nu^m := \lim_{\alpha \in \Lambda} \nu_\alpha^m = \inf_{\alpha \in \Lambda} \nu_\alpha^m.$$

Then, as above, we can choose an increasing sequence $(\alpha_k^m)_{k \in \mathbb{N}}$ in Λ so that

$$\nu^m \leqslant \nu_{\alpha_k}^m < \nu^m + \frac{1}{k},$$
$$\nu^m := \lim_{\alpha \in \Lambda} \nu_\alpha^m = \lim_{k \in \mathbb{N}} \nu_{\alpha_k^m}^m,$$

and for $\alpha \in \Lambda$ with $\alpha \geqslant \alpha_k^m$,

$$\nu^m \leqslant \nu_\alpha^m \leqslant \nu_{\alpha_k^m}^m < \nu^m + \frac{1}{k}.$$

Since (3.1.1) is obvious when M is finite, we may assume M is infinite. Since the set Λ is directed, we can choose $t_1 \in \Lambda$ so that $t_1 \geqslant \alpha_1, \alpha_1^1$. For $k \geqslant 2$, choose $t_k \geqslant t_{k-1}$, α_k, α_k^i, $i = 1, \ldots, k$. Then $(t_k)_{k \in \mathbb{N}}$ is an increasing sequence and for each $m \in M$,

$$\rho = \lim_{k\mathbb{N}} \rho_{t_k} \quad \text{and} \quad \nu^m = \lim_{k \in \mathbb{N}} \nu_{t_k}^m.$$

Let $\varepsilon > 0$ and let $m \in M$ be fixed. Choose k_0 so large that $k_0 > m$ and $\frac{1}{k_o} < \varepsilon$. Then for all $k > k_0$, $t_k > t_{k_0} \geqslant \alpha_{k_0}$, and so

$$\rho \leqslant \rho_{t_k} \leqslant \rho_{t_{k_0}} \leqslant \rho_{\alpha_{k_0}} < \rho + \frac{1}{k_0} < \rho + \varepsilon;$$

and since $t_k > t_{k_0} \geqslant \alpha_{k_0}^m$,

$$\nu^m \leqslant \nu^m_{t_k} \leqslant \nu^m_{t_{k_0}} \leqslant \nu^m_{\alpha^m_{k_0}} < \nu^m + \frac{1}{k_0} < \nu^m + \varepsilon.$$

Thus we may assume we have a decreasing sequence $\{D_n\colon n \geqslant 1\}$ of bounded sets.

Choose $\psi_n \in D_n$ such that

$$\limsup_n \|\phi - \psi_n\| = \limsup_n \bigl\{\|\phi - \psi\|\colon \psi \in D_n\bigr\}.$$

It follows that it suffices to prove the following inequality:

$$\limsup_n \|\phi - \psi_n\| + \limsup_m \|\phi_m - \phi\| \leqslant \limsup_m \limsup_n \|\phi_m - \psi_n\|.$$

We may assume, without loss of generality, that $\phi = 0$ and that $\lim\|\psi_n\|$, $q := \lim\|\phi_m\|$, and $r := \lim_m \limsup_n \|\phi_m - \psi_n\|$ exist. Suppose, on the contrary, that for some $p > 0$ we have

$$\lim\|\psi_n\| = r - q + p. \tag{3.1.3}$$

We will show that for each $\varepsilon > 0$ we can find two sequences $N_1 < N_2 < \cdots$ and finite subsets $\sigma_1 \subset \sigma_2 \subset \cdots$ of I such that for $n \geqslant N_k$,

$$\sum_{i \in \sigma_k \setminus \sigma_{k-1}} \bigl\|\psi_n(i)\bigr\| > (p - \varepsilon)/2, \quad \text{with } \sigma_0 = \emptyset.$$

This would contradict the boundedness of (ψ_n) because for $n \geqslant N_k$,

$$\|\psi_n\| > \sum_{i \in \sigma_k} \bigl\|\psi_n(i)\bigr\| \geqslant k(p - \varepsilon)/2.$$

To show how the two sequences can be constructed, let $\varepsilon > 0$ be arbitrary. Using (3.1.3), there exist an $m_1 \in M$ and an N_1 such that for all $n \geqslant N_1$,

$$\begin{aligned} \|\phi_{m_1}\| &> q - \varepsilon/4; \\ \|\phi_{m_1} - \psi_n\| &< r + \varepsilon/4; \\ \|\psi_n\| &> r - q + p - \varepsilon/4. \end{aligned}$$

For this m_1 choose a finite set $\sigma_1 \subset I$ such that

$$\sum_{i \notin \sigma_1} \bigl\|\phi_{m_1}(i)\bigr\| < \varepsilon/8.$$

Then for all $n \geqslant N_1$,

$$\begin{aligned}
r+\varepsilon/4 &> \|\phi_{m_1}-\psi_n\| \\
&= \sum_{i\in\sigma_1}\|\phi_{m_1}(i)-\psi_n(i)\| + \sum_{i\notin\sigma_1}\|\phi_{m_1}(i)-\psi_n(i)\| \\
&\geqslant \sum_{i\in\sigma_1}\big(\|\phi_{m_1}(i)\|-\|\psi_n(i)\|\big) + \sum_{i\notin\sigma_1}\big(\|\psi_n(i)\|-\|\phi_{m_1}(i)\|\big) \\
&= \|\phi_{m_1}\| - 2\sum_{i\notin\sigma_1}\|\phi_{m_1}(i)\| + \|\psi_n\| - 2\sum_{i\in\sigma_1}\|\psi_n(i)\| \\
&> q-\varepsilon/4-\epsilon/4+r-q+p-\varepsilon/4-2\sum_{i\in\sigma_1}\|\psi_n(i)\| \\
&= r+p-3\varepsilon/4-2\sum_{i\in\sigma_1}\|\psi_n(i)\|.
\end{aligned}$$

Thus

$$\sum_{i\in\sigma_1}\|\psi_n(i)\| > (p-\varepsilon)/2.$$

Next, since (ϕ_m) converges to 0 in the weak* topology, it follows from (3.1.2) that for each $i\in I$, $(\phi_m(i))$ is weak* convergent to 0 in $\mathcal{T}(\mathfrak{H}_i)$, and as $\mathfrak{H}_i$ is finite-dimensional, it is norm-convergent to 0. Using this, we can find an $m_2\in M$ and an $N_2>N_1$ such that for all $n\geqslant N_2$,

$$\begin{aligned}
\sum_{i\in\sigma_1}\|\phi_{m_2}(i)\| &< \varepsilon/10; \\
\|\phi_{m_2}\| &> q-\varepsilon/5; \\
\|\psi_n-\phi_{m_2}\| &< r+\varepsilon/5; \\
\|\psi_n\| &> r-q+p-\varepsilon/5.
\end{aligned}$$

For this m_2, we can find a finite subset $\sigma_2\supset\sigma_1$ such that

$$\sum_{i\notin\sigma_2}\|\phi_{m_2}(i)\| < \varepsilon/10.$$

Then for all $n\geqslant N_2$,

$$\begin{aligned}
r+\varepsilon/5 &> \|\phi_{m_2}-\psi_n\| \\
&= \sum_{i\in\sigma_1}\|\phi_{m_2}(i)-\psi_n(i)\| + \sum_{i\in\sigma_2\setminus\sigma_1}\|\phi_{m_2}(i)-\psi_n(i)\| + \sum_{i\notin\sigma_2}\|\phi_{m_2}(i)-\psi_n(i)\| \\
&\geqslant \sum_{i\in\sigma_1}\big(\|\psi_n(i)\|-\|\phi_{m_2}(i)\|\big) + \sum_{i\in\sigma_2\setminus\sigma_1}\big(\|\phi_{m_2}(i)\|-\|\psi_n(i)\|\big) \\
&\quad + \sum_{i\notin\sigma_2}\big(\|\psi_n(i)\|-\|\phi_{m_2}(i)\|\big)
\end{aligned}$$

$$= \|\phi_{m_2}\| - 2\sum_{i\in\sigma_1} \|\phi_{m_2}(i)\| - 2\sum_{i\notin\sigma_2} \|\phi_{m_2}(i)\| + \|\psi_n\| - 2\sum_{i\in\sigma_2\setminus\sigma_1} \|\psi_n(i)\|$$

$$> q - \varepsilon/5 - \varepsilon/5 - \varepsilon/5 + r - q + p - \varepsilon/5 - 2\sum_{i\in\sigma_2\setminus\sigma_1} \|\psi_n(i)\|$$

$$= r + p - 4\varepsilon/5 - 2\sum_{i\in\sigma_2\setminus\sigma_1} \|\psi_n(i)\|.$$

Thus

$$\sum_{i\in\sigma_2\setminus\sigma_1} \|\psi_n(i)\| > (p-\varepsilon)/2.$$

We repeat the above steps to find an $m_3 \in M$, an $N_3 > N_2$ and a finite subset $\sigma_3 \supset \sigma_2$ such that for all $n \geqslant N_3$,

$$\sum_{i\in\sigma_2} \|\phi_{m_3}(i)\| < \varepsilon/10;$$

$$\sum_{i\notin\sigma_3} \|\phi_{m_3}(i)\| < \epsilon/10;$$

$$\|\phi_{m_3}\| > q - \varepsilon/5;$$

$$\|\psi_n - \phi_{m_3}\| < r + \varepsilon/5;$$

$$\|\psi_n\| > r - q + p - \varepsilon/5.$$

From these inequalities we obtain, as before,

$$\sum_{i\in\sigma_3\setminus\sigma_2} \|\psi_n(i)\| > (p-\varepsilon)/2.$$

We continue this process to obtain the desired sequences. □

Corollary 3.2. *Suppose G is a compact group. If (ψ_n) is a bounded net in $B(G)$, and (ϕ_m) is a sequence that converges to ϕ in the weak* topology, then*

$$\limsup_n \|\psi_n - \phi\| + \limsup_m \|\phi_m - \phi\| = \limsup_m \limsup_n \|\psi_n - \phi_m\|.$$

A dual Banach space E is said to have the lim–sup property if whenever (ϕ_n) is a sequence in E that converges to 0 in the weak* topology and $\lim_n \|\phi_n\|$ exists, then $\lim_n \|\phi_n - \psi\| = \lim_n \|\phi_n\| + \|\phi - \psi\|$ for any $\psi \in E$. In [35], Lim showed that ℓ_1 has this property and as a consequence, ℓ_1 has weak* normal structure.

Corollary 3.3. *Let G be a compact group. If (ϕ_m) is a sequence in $B(G)$ that converges to ϕ in the weak* topology, then for all $\psi \in B(G)$,*

$$\limsup_m \|\phi_m - \psi\| = \|\psi - \phi\| + \limsup_m \|\phi_m - \phi\|.$$

In addition, if $\lim_m \|\phi_m - \phi\|$ *exists, then*

$$\lim_m \|\phi_m - \psi\| = \lim_m \|\phi_m - \phi\| + \|\phi - \psi\|.$$

In particular, $B(G)$ *has the lim–sup property.*

4. Weak* fixed point property for left reversible semigroups

We are now ready to state and prove our results for $B(G)$ of a separable compact group G.

Let C be a nonempty subset of a Banach space X and $\{D_\alpha\colon \alpha \in \Lambda\}$ be a decreasing net of bounded nonempty subsets of X. For each $x \in C$, and $\alpha \in \Lambda$, let

$$\begin{aligned} r_\alpha(x) &= \sup\bigl\{\|x - y\|\colon\ y \in D_\alpha\bigr\}, \\ r(x) &= \lim_\alpha r_\alpha(x) = \inf_\alpha r_\alpha(x), \\ r &= \inf\bigl\{r(x)\colon\ x \in C\bigr\}. \end{aligned}$$

The set (possibly empty)

$$\mathcal{AC}\bigl(\{D_\alpha\colon\ \alpha \in \Lambda\}\bigr) = \bigl\{x \in C\colon\ r(x) = r\bigr\}$$

is called the *asymptotic center* of $\{D_\alpha\colon \alpha \in \Lambda\}$ with respect to C and r is called the *asymptotic radius* of $\{D_\alpha\colon \alpha \in \Lambda\}$ with respect to C.

Theorem 4.1. *Let G be a separable compact group. Let C be a nonempty weak* closed convex subset of $B(G)$ and $\{D_\alpha\colon \alpha \in \Lambda\}$ be a decreasing net of nonempty bounded subsets of C. Let $r(x)$ be as defined above. Then for each $s \geqslant 0$, $\{x \in C\colon r(x) \leqslant s\}$ is weak* compact and convex, and the asymptotic center of $\{D_\alpha\colon \alpha \in \Lambda\}$ with respect to C is a nonempty norm compact convex subset of C.*

Proof. First, we observe that since G is separable, the group C^*-algebra $C^*(G)$ is separable, and so the weak* topology on bounded subsets of $B(G)$ is metrizable. Next, we show the convex function $r(\phi)$ is weak* lower semi-continuous. To this end, it suffices to prove the level set $K_s := \{x \in C\colon r(x) \leqslant s\}$ is weak* closed for each s. We may assume that $s \geqslant 0$. Let (ψ_m) be a sequence in K_s which converges to ψ in the weak* topology. By Lemma 3.1

$$r(\psi) = \limsup_m \ r(\psi_m) - \limsup_m \|\psi_m - \psi\| \leqslant s.$$

Hence $\psi \in K_s$, and K_s is weak* closed.

Now denote the asymptotic centre by K and the asymptotic radius by r. For $s > r$, let $r_0 := \inf\{r(x)\colon x \in C \cap K_s\}$ and $K_0 = \{x \in C \cap K_s\colon r(x) = r_0\}$. We have $r = r_0$ and $K = K_0$. Now, the set K_s is norm bounded and weak* closed, so it is weak* compact. Thus the weak* lower semi-continuous convex functional must attain its minimum on the set $C \cap K_s$. Hence $K \neq \emptyset$.

Next, we prove K compact. Suppose (ϕ_n) is a sequence in K that converges to ϕ in the weak* topology. Then $r(\phi_n) = r(\phi) = r$ and, from Lemma 3.1, we must have $\limsup_n \|\phi_n - \phi\| = 0$. Thus the sequence is norm convergent; hence K is compact. $\square$

Theorem 4.2. *Let G be a separable compact group. Then $B(G)$ has the weak* fixed point property for left reversible semigroups.*

Proof. Let S be a left reversible semitopological semigroup, and C be a weak* compact convex nonempty subset of $B(G)$ for which the action of S on $(C, \|\cdot\|)$ is separately continuous and nonexpansive. Let S be directed by $a \geqslant b$ iff $aS \subseteq \overline{bS}$. For a fixed $u \in C$, let $W_s = \overline{sS(u)}$ for all $s \in S$. Then $\{W_s\colon s \in S\}$ is a decreasing net of subsets of C. Let K be the asymptotic center of $\{W_s\colon s \in S\}$ with respect to C. By Theorem 4.1, K is a nonempty compact convex subset of C. Moreover, it is S-invariant. For, let $x \in K$, $s \in S$, and $\epsilon > 0$ be arbitrary. Since $x \in K$, there exists $t \in S$ such that $tS(u) \subset W_t \subset B[x, r + \epsilon]$, where r is the asymptotic radius and $B[x, r]$ denotes the closed ball of radius r centered at x. Since s is nonexpansive, we have $stS(u) \subset B[s(x), r + \epsilon]$, so that $W_{st} \subset B[s(x), r + \epsilon]$. Thus, $s(x) \in K$. It now follows from Corollary 1 in [18] that K, and hence C, contains a common fixed point for S. $\square$

Remark 4.3. For a locally compact group G, denote the set of equivalence classes of irreducible unitary representation of G by $\widehat{G}$. When $G = T$, the circle group, then $\widehat{G}$ is the dual group isomorphic to the integers $\mathbb{Z}$. In this case, as is well known, $C^*(G)$ is isometric isomorphic to $c_0(\mathbb{Z})$ via the Fourier transform, and $B(G)$ is isometric isomorphic to $\ell_1(\mathbb{Z})$ via Bochner's Theorem. See Examples 1.9 and 2.5 in [13]. Thus our Theorem 4.2 can be seen to be a generalization of Lim's result that ℓ_1 has the weak* fixed point property for left reversible semigroups.

Remark 4.4. We were not able to remove separability from the hypothesis of Theorem 4.2. As far as we know, it is even unknown whether $\ell_1(\Gamma)$ has the weak* fixed point property for left reversible semigroups when Γ is uncountable.

Remark 4.5. Let G be a separable locally compact group. Then the measure algebra $M(G)$ has the weak* fixed point property for left reversible semigroups if and only if G is discrete. If $M(G)$ has the weak* fixed point property for left reversible semigroups then it has the weak* fixed point property, and so G must be discrete by [26, Theorem 1]. The other direction follows from Lim's result [35, Theorem 4].

A locally compact group G is called an [IN]-group if there is a compact neighbourhood of the identity e in G which is invariant under inner automorphisms. The class of [IN]-group contains all discrete groups, abelian groups and compact groups. Every [IN]-group is unimodular.

Theorem 4.6. *Let G be a separable* [IN]*-group. Then the following are equivalent*:

(a) *G is compact.*
(b) *$B(G)$ has property* UKK*.
(c) *$B(G)$ has weak* normal structure.*
(d) *$B(G)$ has weak* fixed point property for left reversible semigroups.*

(e) *$B(G)$ has the weak* fixed point property.*
(f) *$B(G)$ has the lim–sup property.*
(g) *$B(G)$ is separable.*
(h) *$\widehat{G}$ is countable.*

Proof. (a) $\Rightarrow$ (b) was proved in [26].

(b) $\Rightarrow$ (c) was proved in [11].

(c) $\Rightarrow$ (e) was in proved in [35].

(e) $\Rightarrow$ (a) Since every weakly compact convex subset of $A(G)$ is weakly compact convex in $B(G)$, it follows that it is a weak* compact convex subset of $B(G)$. Thus by assumption (e), $A(G)$ has the weak fixed point property. Since G is an [IN]-group, it follows from [24, Corollary 4.2] that G is compact.

(a) $\Rightarrow$ (f) follows from Corollary 3.3.

(f) $\Rightarrow$ (c) If G is separable, then, as we observe in the proof of Lemma 3.1, $C^*(G)$ is separable. If in Theorem 2 in [35] Lim defines the function δ by $\delta(r,s) = r + s$, then it is easy to see that δ satisfies conditions (i) and (ii), and by (f), δ satisfies (iii). And so it follows from that theorem that $B(G)$ has weak* normal structure.

(a) $\Rightarrow$ (d) follows from Theorem 4.2, and (d) $\Rightarrow$ (e) is trivial.

Finally, the equivalence of (g) and (h) to the compactness of G follows from Theorem 6.1 and Lemma 6.2 in [17]. $\square$

Remark 4.7. (a) If G is separable then so is $A(G)$, and conversely. See [17, Corollary 6.9].

(b) There is a non-compact locally compact group G, the so-called Fell's group, for which $\widehat{G}$ is countable. See [2]. The Fell's group G is the semi-direct product of the additive p-adic number field Q_p and the multiplicative compact group of p-adic units for a fixed prime p. So G is solvable and hence amenable. The unit ball of $B(G)$ is weak* sequentially compact. So by [26, Theorem 5], $B(G)$ cannot have property UKK*. By Proposition 5.1 in Section 5, $B(G)$ has the weak fixed point property for left reversible semigroups. However, it is unknown whether $B(G)$ even has the weak* fixed point property.

(c) It is known that if G is an [IN]-group, then G is compact iff $B(G)$ has the Radon–Nikodym property iff $B(G)$ has the Krein–Milman property. See [12,16,24].

5. Weak fixed point property for a semigroup

We now investigate the weak fixed point property for a semigroup. A group G is said to be an [AU]-group if the von Neumann algebra generated by every continuous unitary representation of G is atomic. It is an [AR]-group if the von Neumann algebra $VN(G)$ is atomic. Since $VN(G)$ is the von Neumann algebra generated by the regular representation, it is clear that every [AU]-group is an [AR]-group. It was shown in [27, Lemma 3.1] that if the predual $\mathfrak{M}_*$ of a von Neumann algebra $\mathfrak{M}$ has the Radon–Nikodym property, then $\mathfrak{M}_*$ has the weak fixed point property. In fact, since the property UKK is hereditary, the proof there actually showed $\mathfrak{M}_*$ has property UKK, and hence has weak normal structure. For the two preduals $A(G)$ and $B(G)$, we know from [38, Theorem 4.1 and Theorem 4.2] that the class of groups for which $A(G)$ and $B(G)$ have the Radon–Nikodym property are precisely the [AR]-groups and [AU]-groups, respectively. Thus by Lim's result [34, Theorem 3] we have

Proposition 5.1. *Let G be a locally compact group.*

(a) *If G is an* [AR]*-group, then $A(G)$ has the weak fixed point property for left reversible semigroups.*
(b) *If G is an* [AU]*-group, then $B(G)$ has the weak fixed point property for left reversible semigroups.*

If [compact] denote the class of compact groups, etc., then we have the inclusions [compact] $\subset$ [AU] $\subset$ [AR], so that $A(G)$ and $B(G)$ have the weak fixed point property for left reversible semigroups when G is compact. Moreover, the inclusions are proper. For example, if G is the Fell's group, then G is a non-compact group for which $B(G)$ (and hence $A(G)$) has the weak fixed point property for left reversible semigroups. See [38, Remark 4.6].

In view of [24, Corollary 4.2], we have the following for [IN]-groups.

Proposition 5.2. *Let G be an* [IN]*-group. Then the following are equivalent*:

(a) *G is compact.*
(b) *$A(G)$ has property* UKK.
(c) *$A(G)$ has weak normal structure.*
(d) *$A(G)$ has the weak fixed point property for left reversible semigroups.*
(e) *$A(G)$ has the weak fixed point property.*
(f) *$A(G)$ has the Radon–Nikodym property.*
(g) *$A(G)$ has the Krein–Milman property.*

Proposition 5.3. *Let G be a locally compact group. Then the group algebra $L^1(G)$ has the weak fixed point property for left reversible semigroups if and only if G is discrete.*

Proof. If $L^1(G)$ has the weak fixed point property for left reversible semigroup, then it has the weak fixed point property. If G is not discrete, then $L^1(G)$ contains an isometric copy of $L^1[0,1]$ (see [21, p. 136]), which contradicts Alspach's result in [1].

Conversely, if G is discrete, then $L^1(G)$ has weak* normal structure [26, Theorem 1]. It follows from [34, Theorem 3] that $L^1(G)$ has the weak fixed point property for left reversible semigroups. □

It is well known that the weak fixed point property is separably determined, i.e., a Banach space X has the weak fixed point property if and only if all its separable closed subspaces do. See [15, p. 35]. In [14], García-Falset defined the coefficient $R(X)$ of a Banach space X by

$$R(X) := \sup\Big\{\liminf_{n\to\infty}\|x_n + x\|\Big\},$$

where the supremum is taken over all weakly null sequences (x_n) of the unit ball and all points x of the unit ball, and then showed that any Banach space X with $R(X) < 2$ has the weak fixed point property. In a private communication he has informed us that he has shown that if X^* has the UKK* property and the unit ball is weak* sequentially compact, then $R(X) < 2$. The authors would like to express our gratitude to him for sending us the proof of this fact. Now for $\mathcal{K}(\mathfrak{H}_0)$, the compact operators on an arbitrary Hilbert space $\mathfrak{H}_0$, Lennard proved in [33] its dual has the UKK* property. It follows from the results of García-Falset mentioned above that if, in addition, $\mathfrak{H}_0$ is separable, then $R(\mathcal{K}(\mathfrak{H}_0)) < 2$, and so $\mathcal{K}(\mathfrak{H}_0)$ has the weak fixed point property. Alternatively, this fact also follows from [10, Theorem 3]. See also the note on page 775 in [10]. Now if $\mathfrak{H}$ is any Hilbert space, and Y is any separable closed subspace of $\mathcal{K}(\mathfrak{H})$, then as the proof of Theorem 5 in [9] shows, every separable subspace Y of $\mathcal{K}(\mathfrak{H})$ can be embedded as a subspace of $\mathcal{K}(\mathfrak{H}_0)$, where $\mathfrak{H}_0$ is a separable Hilbert subspace of $\mathfrak{H}$. Thus Y and hence $\mathcal{K}(\mathfrak{H})$ has the weak fixed point property. Now let G be a compact group. Then the group C^*-algebra $C^*(G)$ is a C^*-subalgebra of $\mathcal{K}(L^2(G))$. So $C^*(G)$ has the weak fixed point property when G is compact. This answers Questions 3 in [27]. See [7] for results on fixed point property for C^*-algebras and in particular for $C^*(G)$. Combining this result with Bruck's result [5], we have the following

Proposition 5.4. *If G is a compact group, then $C^*(G)$ has the weak fixed point property for commutative semigroups.*

We note that Lim's result cannot be applied here to conclude that $C^*(G)$ has the weak fixed point property for left reversible semigroups since $C^*(G)$ does not have weak normal structure unless G is finite. However, we do not know whether it is possible for $C^*(G)$ to have the weak fixed point property for left reversible semigroups without having weak normal structure. Nor do we know if the converse of Proposition 5.4 is true. However, we do know that G must be an [AU]-group (see [27, Corollary 4.2] and [6, Theorem 3]).

Proposition 5.5. *$VN(G)$ has the weak (weak*) fixed point property for left reversible semigroups if and only if G is finite.*

It was shown in [27, Corollary 4.4] that $VN(G)$ has the weak fixed point property if and only if G is finite. It follows from this that if $VN(G)$ has the weak fixed point property for left reversible semigroups then G must be finite. Conversely, if G is finite, then $VN(G)$ is finite-dimensional, and so it has the weak fixed point property for left reversible semigroups by [18, Corollary 1].

6. Remarks and open problems

Remark 6.1. Lemma 3.1 does not hold for general dual Banach spaces. Suppose, on the contrary, that the conclusion of Lemma 3.1 is true in a dual Banach space E, that is, whenever $\{D_\alpha\colon \alpha \in \Lambda\}$ is a decreasing net of bounded subsets of E, and (ϕ_m) is a weak* convergent sequence with weak* limit ϕ, then (3.1.1) holds. If for all $\alpha \in \Lambda$ we take $D_\alpha = \{\psi\}$ and assume that $\lim_m \|\phi_m - \phi\| = s$, then we would have $\|\psi - \phi\| + s = \limsup_m \|\phi_m - \psi\|$, that is, E satisfies the lim–sup property. The proof of Theorem 5 in [25] showed that $\mathcal{T}(\mathfrak{H})$, the trace-class operators on a Hilbert space $\mathfrak{H}$, does not have the lim–sup property if $\mathfrak{H}$ is infinite-dimensional. Thus we cannot hope to extend Lemma 3.1 to $\mathcal{T}(\mathfrak{H})$ for infinite-dimensional $\mathfrak{H}$.

Open Problem 6.2. Bruck proved in [5] a Banach space E has the weak fixed point property for commuting semigroups if it has the weak fixed point property. If a dual Banach space E has the weak* fixed point property, does E have the weak* fixed point property for commuting semigroups, or left reversible semigroups?

Open Problem 6.3. Lim proved in [34] a Banach space has the weak fixed point property for left reversible semigroups if it has weak normal structure. Let E be a dual Banach space with weak* normal structure. Does E have the weak* fixed point property for left reversible semigroups? In particular, note that as a consequence of the result in [33], $\mathcal{T}(\mathfrak{H})$ has weak* normal structure and hence the weak* fixed point property. Does $\mathcal{T}(\mathfrak{H})$ have the weak* fixed point property for left reversible semigroups?

We have the following result.

Proposition 6.4. *Let E be a dual Banach space which has weak* normal structure. If G is a group of isometric self-maps of a weak* compact convex subset K, then K has a common fixed point for G.*

Proof. An application of Zorn's Lemma gives the existence of a minimal G-invariant nonempty weak* compact convex subset $X \subseteq K$. A second application of Zorn's Lemma yields a minimal G-invariant nonempty weak*-compact subset $M \subseteq X$.

If M is a singleton then we have a common fixed point. Suppose M contains more than one point. First, we show $g(M) = M$ for each $g \in G$. It suffices to show that g is onto. If $m \in M$ is arbitrary, then $m = gg^{-1}(m) \in gM$, provided we can show $e(m) = m$ for the group identity e. For each $g \in G$, g is an isometry and so $\|e(m) - m\| = \|g(em) - g(m)\| = \|g(m) - g(m)\| = 0$. Thus $e(m) = m$, and so $g(M) = M$. Next, since K has weak* normal structure, M contains a point $u \in \text{w}^*\text{-clco}(M) := M_1$ such that

$$\rho := \sup\bigl\{\|u - y\|:\ y \in M\bigr\} < \operatorname{diam}(M_1) = \operatorname{diam}(M).$$

For each $y \in M$, let $Y_y := \{x \in X:\ \|y - x\| \leqslant \rho\}$ and $Y = \bigcap_{y \in M} Y_y$. Then (i) $Y \neq \emptyset$ since $u \in Y$; (ii) Y is weak*-compact and convex since $Y = \bigcap_{y \in M}(X \cap B[y, \rho])$, where $B[y, \rho]$ is the closed ball centred at y with radius ρ; (iii) Y is a proper subset of X since if $X \subseteq Y$ then $M \subseteq X \subseteq B[y, \rho]$, contradicting that $\operatorname{diam}(M) > \rho$. Thus the set Y contradicts the minimality of X. Consequently, M must be a singleton. □

Open Problem 6.5. Let E be a Banach space with the weak fixed point property. Does E have the weak fixed point property for left reversible semigroups?

Open Problem 6.6. If $B(G)$ has any of the properties UKK*, weak* normal structure, the weak* fixed point property, the weak* fixed point property for left reversible semigroups, or the lim–sup property, does it follow that G is compact?

Open Problem 6.7. Is there a fixed point property for groups of isometries on weak* compact convex sets in a dual Banach space which characterizes G-amenability of von Neumann algebras of a locally compact group G as defined in [28]? See also [19].

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